AP CALCULUS AB
SUMMER ASSIGNMENT

Welcome to AP Calculus AB!

Unit 1 for this course is called ***Limits***. Your assignment for this summer is to read the first two lessons of material in this PDF. You should try your best to understand each worked example and you should work through the checkpoint questions.

Lesson 1: A Preview of Calculus
Lesson 2: The Limit of a Function

We will cover ALL of this material in class when school begins, so please do not panic if you have questions. Having a foundation before we begin class will *greatly* improve your understanding!

If you get stuck while working through this packet, please use the internet as a resource. Khan Academy and Paul's Online Math Notes are both great and free resources.

Please email me with any questions. I am looking forward to teaching you next year, and I hope you have a wonderful summer!

-Mrs. Guillory

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The following pages are from a free, open source textbook called OpenStax.
The full textbook can be found at this link: https://openstax.org/details/books/calculus-volume-1

2 | LIMITS

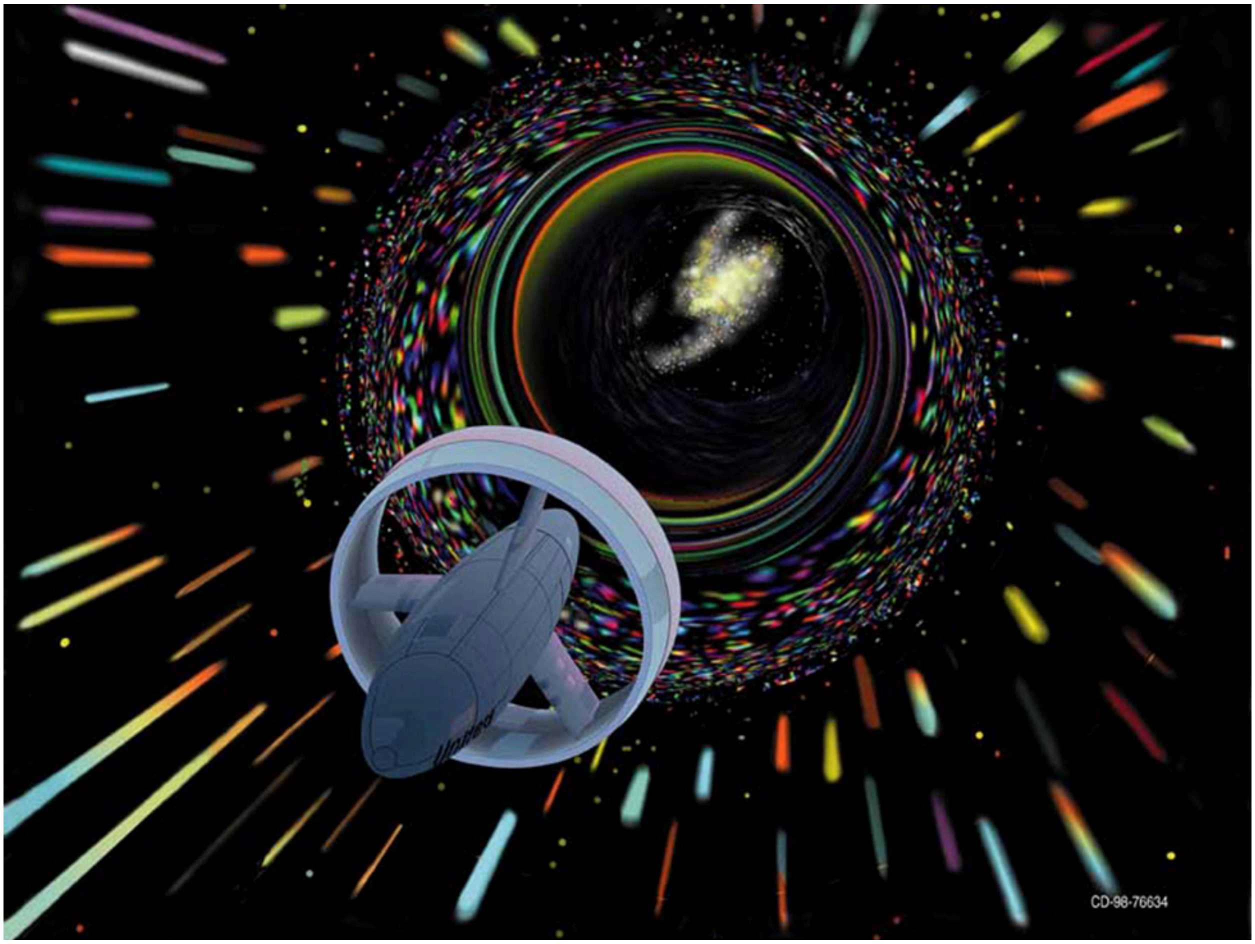


Figure 2.1 The vision of human exploration by the National Aeronautics and Space Administration (NASA) to distant parts of the universe illustrates the idea of space travel at high speeds. But, is there a limit to how fast a spacecraft can go? (credit: NASA)

Chapter Outline

Introduction

Science fiction writers often imagine spaceships that can travel to far-off planets in distant galaxies. However, back in 1905, Albert Einstein showed that a limit exists to how fast any object can travel. The problem is that the faster an object moves, the more mass it attains (in the form of energy), according to the equation

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. What is this speed limit? (We explore this problem further in **Example 2.12**.)

The idea of a limit is central to all of calculus. We begin this chapter by examining why limits are so important. Then, we go on to describe how to find the limit of a function at a given point. Not all functions have limits at all points, and we discuss what this means and how we can tell if a function does or does not have a limit at a particular value. This chapter has been created in an informal, intuitive fashion, but this is not always enough if we need to prove a mathematical statement involving limits. The last section of this chapter presents the more precise definition of a limit and shows how to prove whether a function has a limit.

2.1 | A Preview of Calculus

Learning Objectives

2.1.1 Describe the tangent problem and how it led to the idea of a derivative.
2.1.2 Explain how the idea of a limit is involved in solving the tangent problem.
2.1.3 Recognize a tangent to a curve at a point as the limit of secant lines.
2.1.4 Identify instantaneous velocity as the limit of average velocity over a small time interval.
2.1.5 Describe the area problem and how it was solved by the integral.
2.1.6 Explain how the idea of a limit is involved in solving the area problem.
2.1.7 Recognize how the ideas of limit, derivative, and integral led to the studies of infinite series and multivariable calculus.

As we embark on our study of calculus, we shall see how its development arose from common solutions to practical problems in areas such as engineering physics—like the space travel problem posed in the chapter opener. Two key problems led to the initial formulation of calculus: (1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point; and (2) the area problem, or how to determine the area under a curve.

The Tangent Problem and Differential Calculus

Rate of change is one of the most critical concepts in calculus. We begin our investigation of rates of change by looking at the graphs of the three lines $f(x) = -2x - 3$, $g(x) = \frac{1}{2}x + 1$, and $h(x) = 2$, shown in **Figure 2.2**.

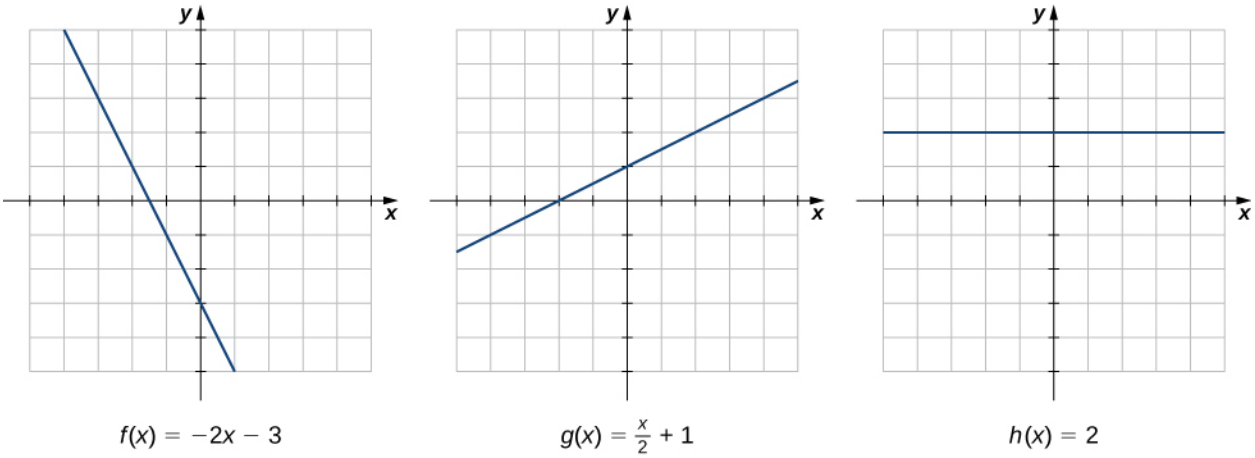


Figure 2.2 The rate of change of a linear function is constant in each of these three graphs, with the constant determined by the slope.

As we move from left to right along the graph of $f(x) = -2x - 3$, we see that the graph decreases at a constant rate. For every 1 unit we move to the right along the x-axis, the y-coordinate decreases by 2 units. This rate of change is determined by the slope (−2) of the line. Similarly, the slope of 1/2 in the function $g(x)$ tells us that for every change in x of 1 unit there is a corresponding change in y of 1/2 unit. The function $h(x) = 2$ has a slope of zero, indicating that the values of the function remain constant. We see that the slope of each linear function indicates the rate of change of the function.

Compare the graphs of these three functions with the graph of $k(x) = x^2$ (**Figure 2.3**). The graph of $k(x) = x^2$ starts from the left by decreasing rapidly, then begins to decrease more slowly and level off, and then finally begins to increase—slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function?

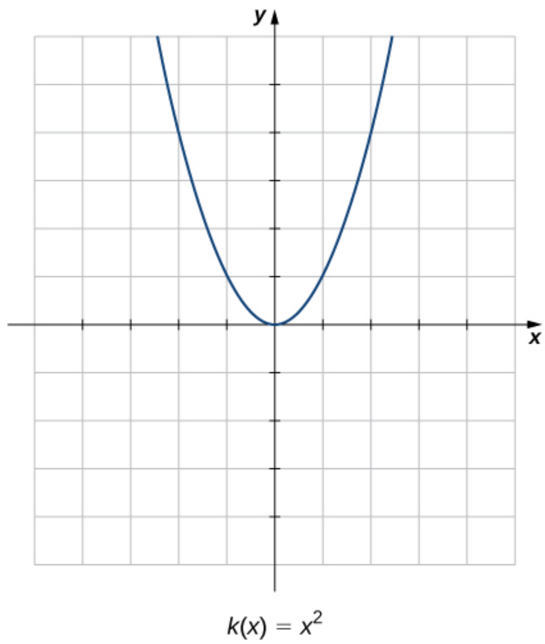


Figure 2.3 The function $k(x) = x^2$ does not have a constant rate of change.

We can approximate the rate of change of a function $f(x)$ at a point $(a, f(a))$ on its graph by taking another point $(x, f(x))$ on the graph of $f(x)$, drawing a line through the two points, and calculating the slope of the resulting line. Such a line is called a **secant** line. **Figure 2.4** shows a secant line to a function $f(x)$ at a point $(a, f(a))$.

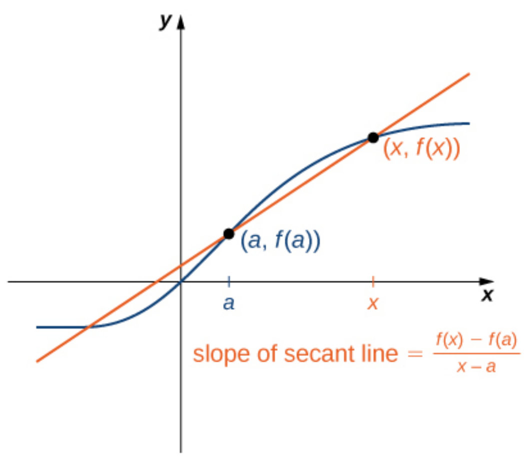


Figure 2.4 The slope of a secant line through a point $(a, f(a))$ estimates the rate of change of the function at the point $(a, f(a))$.

We formally define a secant line as follows:

Definition

The **secant** to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}. \tag{2.1}$$

The accuracy of approximating the rate of change of the function with a secant line depends on how close x is to a. As we see in **Figure 2.5**, if x is closer to a, the slope of the secant line is a better measure of the rate of change of $f(x)$ at a.

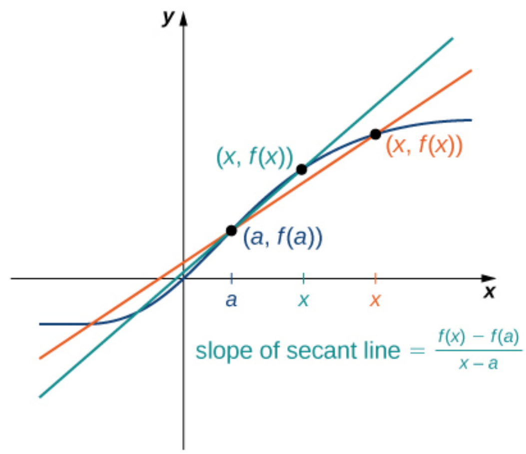


Figure 2.5 As x gets closer to a, the slope of the secant line becomes a better approximation to the rate of change of the function $f(x)$ at a.

The secant lines themselves approach a line that is called the **tangent** to the function $f(x)$ at a (**Figure 2.6**). The slope of the tangent line to the graph at a measures the rate of change of the function at a. This value also represents the derivative of the function $f(x)$ at a, or the rate of change of the function at a. This derivative is denoted by $f'(a)$. **Differential calculus** is the field of calculus concerned with the study of derivatives and their applications.

For an interactive demonstration of the slope of a secant line that you can manipulate yourself, visit this applet (*Note:* this site requires a Java browser plugin): **Math Insight (http://www.openstax.org/l/20_mathinsight)**.

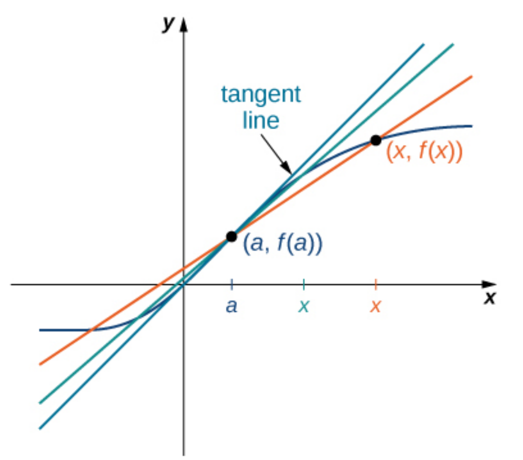


Figure 2.6 Solving the Tangent Problem: As x approaches a, the secant lines approach the tangent line.

Example 2.1 illustrates how to find slopes of secant lines. These slopes estimate the slope of the tangent line or, equivalently, the rate of change of the function at the point at which the slopes are calculated.

Example 2.1

Finding Slopes of Secant Lines

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and each of the following points on the graph of $f(x) = x^2$.

a. $(2, 4)$

b. $\left(\frac{3}{2}, \frac{9}{4}\right)$

Solution

Use the formula for the slope of a secant line from the definition.

a. $m_{\text{sec}} = \frac{4 - 1}{2 - 1} = 3$

b. $m_{\text{sec}} = \frac{\frac{9}{4} - 1}{\frac{3}{2} - 1} = \frac{5}{2} = 2.5$

The point in part b. is closer to the point $(1, 1)$, so the slope of 2.5 is closer to the slope of the tangent line. A good estimate for the slope of the tangent would be in the range of 2 to 2.5 (**Figure 2.7**).

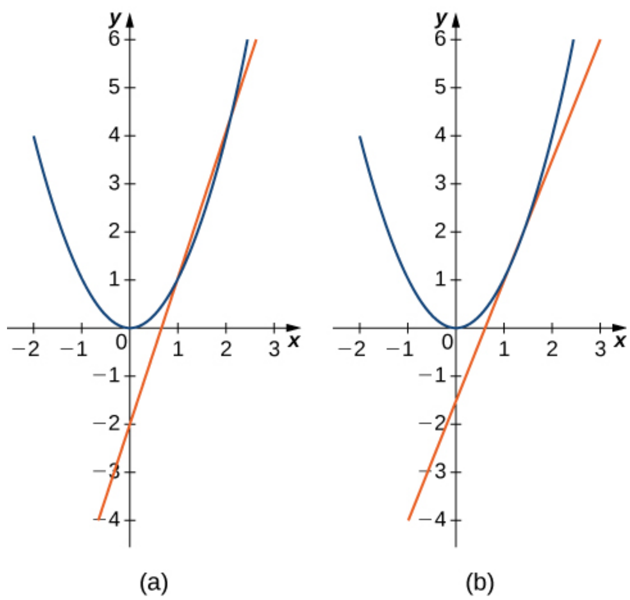


Figure 2.7 The secant lines to $f(x) = x^2$ at $(1, 1)$ through (a) $(2, 4)$ and (b) $\left(\frac{3}{2}, \frac{9}{4}\right)$ provide successively closer approximations to the tangent line to $f(x) = x^2$ at $(1, 1)$.

2.1 Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and the point $\left(\frac{5}{4}, \frac{25}{16}\right)$ on the graph of $f(x) = x^2$.

We continue our investigation by exploring a related question. Keeping in mind that velocity may be thought of as the rate of change of position, suppose that we have a function, $s(t)$, that gives the position of an object along a coordinate axis at any given time *t*. Can we use these same ideas to create a reasonable definition of the instantaneous velocity at a given time $t = a$? We start by approximating the instantaneous velocity with an average velocity. First, recall that the speed of an object traveling at a constant rate is the ratio of the distance traveled to the length of time it has traveled. We define the **average velocity** of an object over a time period to be the change in its position divided by the length of the time period.

Definition

Let $s(t)$ be the position of an object moving along a coordinate axis at time *t*. The **average velocity** of the object over a time interval $[a, t]$ where $a < t$ (or $[t, a]$ if $t < a$) is

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}. \quad (2.2)$$

As *t* is chosen closer to *a*, the average velocity becomes closer to the instantaneous velocity. Note that finding the average velocity of a position function over a time interval is essentially the same as finding the slope of a secant line to a function. Furthermore, to find the slope of a tangent line at a point *a*, we let the *x*-values approach *a* in the slope of the secant line. Similarly, to find the instantaneous velocity at time *a*, we let the *t*-values approach *a* in the average velocity. This process of letting *x* or *t* approach *a* in an expression is called taking a **limit**. Thus, we may define the **instantaneous velocity** as follows.

Definition

For a position function $s(t)$, the **instantaneous velocity** at a time $t = a$ is the value that the average velocities approach on intervals of the form $[a, t]$ and $[t, a]$ as the values of *t* become closer to *a*, provided such a value exists.

Example 2.2 illustrates this concept of limits and average velocity.

Example 2.2

Finding Average Velocity

A rock is dropped from a height of 64 ft. It is determined that its height (in feet) above ground *t* seconds later (for $0 \le t \le 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

a. $[0.49, 0.5]$

b. $[0.5, 0.51]$

Solution

Substitute the data into the formula for the definition of average velocity.

a. $v_{\text{ave}} = \frac{s(0.5) - s(0.49)}{0.5 - 0.49} = -15.84$

b. $v_{\text{ave}} = \dfrac{s(0.51) - s(0.5)}{0.51 - 0.5} = -16.16$

The instantaneous velocity is somewhere between −15.84 and −16.16 ft/sec. A good guess might be −16 ft/sec.

2.2 An object moves along a coordinate axis so that its position at time t is given by $s(t) = t^3$. Estimate its instantaneous velocity at time $t = 2$ by computing its average velocity over the time interval $[2, 2.001]$.

The Area Problem and Integral Calculus

We now turn our attention to a classic question from calculus. Many quantities in physics—for example, quantities of work—may be interpreted as the area under a curve. This leads us to ask the question: How can we find the area between the graph of a function and the *x*-axis over an interval (**Figure 2.8**)?

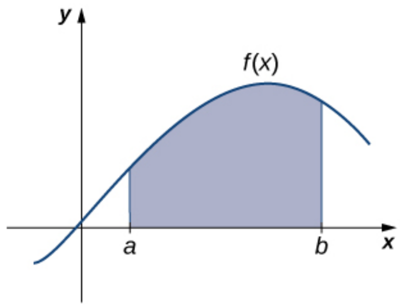


Figure 2.8 The Area Problem: How do we find the area of the shaded region?

As in the answer to our previous questions on velocity, we first try to approximate the solution. We approximate the area by dividing up the interval $[a, b]$ into smaller intervals in the shape of rectangles. The approximation of the area comes from adding up the areas of these rectangles (**Figure 2.9**).

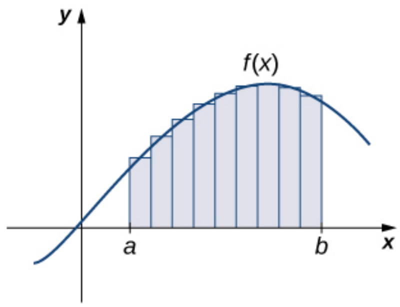


Figure 2.9 The area of the region under the curve is approximated by summing the areas of thin rectangles.

As the widths of the rectangles become smaller (approach zero), the sums of the areas of the rectangles approach the area between the graph of $f(x)$ and the *x*-axis over the interval $[a, b]$. Once again, we find ourselves taking a limit. Limits of this type serve as a basis for the definition of the definite integral. **Integral calculus** is the study of integrals and their applications.

Example 2.3

Estimation Using Rectangles

Estimate the area between the x-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown in **Figure 2.10**.

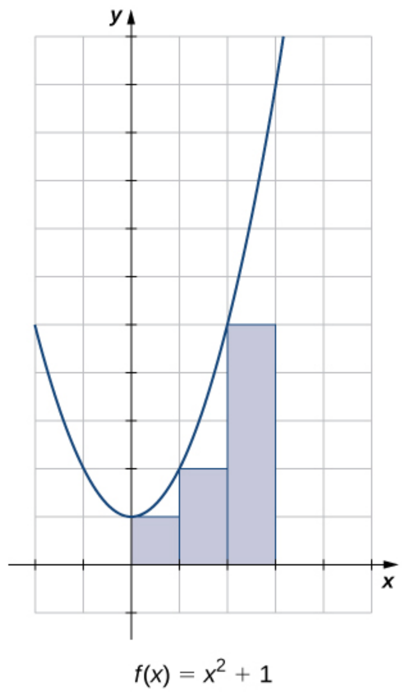
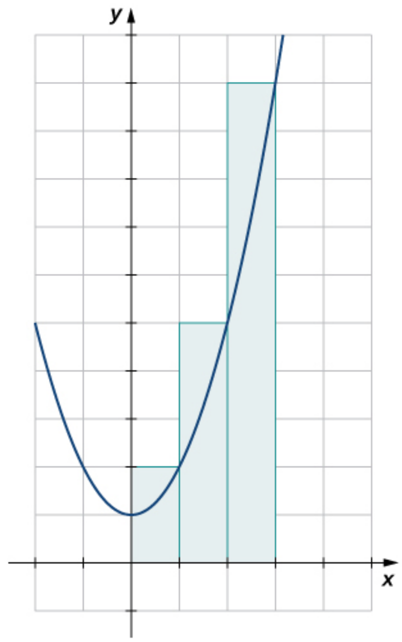


Figure 2.10 The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.

Solution

The areas of the three rectangles are 1 unit^2, 2 unit^2, and 5 unit^2. Using these rectangles, our area estimate is 8 unit^2.

2.3 Estimate the area between the x-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown here:



Other Aspects of Calculus

So far, we have studied functions of one variable only. Such functions can be represented visually using graphs in two dimensions; however, there is no good reason to restrict our investigation to two dimensions. Suppose, for example, that instead of determining the velocity of an object moving along a coordinate axis, we want to determine the velocity of a rock fired from a catapult at a given time, or of an airplane moving in three dimensions. We might want to graph real-value functions of two variables or determine volumes of solids of the type shown in **Figure 2.11**. These are only a few of the types of questions that can be asked and answered using **multivariable calculus**. Informally, multivariable calculus can be characterized as the study of the calculus of functions of two or more variables. However, before exploring these and other ideas, we must first lay a foundation for the study of calculus in one variable by exploring the concept of a limit.

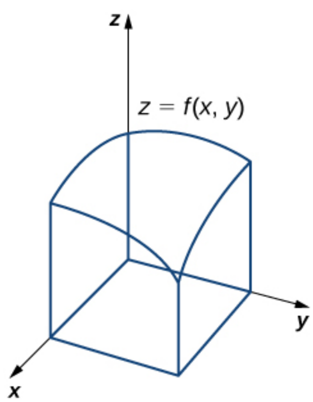


Figure 2.11 We can use multivariable calculus to find the volume between a surface defined by a function of two variables and a plane.

2.2 | The Limit of a Function

Learning Objectives

2.2.1 Using correct notation, describe the limit of a function.
2.2.2 Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
2.2.3 Use a graph to estimate the limit of a function or to identify when the limit does not exist.
2.2.4 Define one-sided limits and provide examples.
2.2.5 Explain the relationship between one-sided and two-sided limits.
2.2.6 Using correct notation, describe an infinite limit.
2.2.7 Define a vertical asymptote.

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach. At the end of this chapter, armed with a conceptual understanding of limits, we examine the formal definition of a limit.

We begin our exploration of limits by taking a look at the graphs of the functions

$$f(x) = \frac{x^2 - 4}{x - 2}, \; g(x) = \frac{|x - 2|}{x - 2}, \; \text{and } h(x) = \frac{1}{(x - 2)^2},$$

which are shown in **Figure 2.12**. In particular, let's focus our attention on the behavior of each graph at and around $x = 2$.

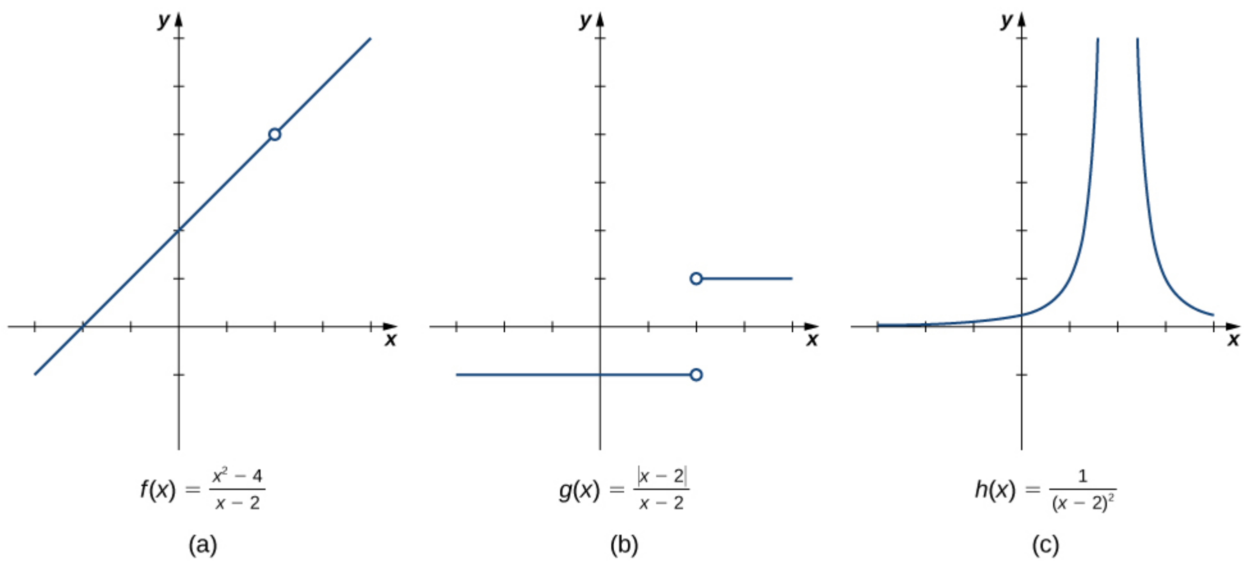


Figure 2.12 These graphs show the behavior of three different functions around $x = 2$.

Each of the three functions is undefined at $x = 2$, but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of $x = 2$. To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.

Intuitive Definition of a Limit

Let's first take a closer look at how the function $f(x) = (x^2 - 4)/(x - 2)$ behaves around $x = 2$ in **Figure 2.12**. As the values of x approach 2 from either side of 2, the values of $y = f(x)$ approach 4. Mathematically, we say that the limit of $f(x)$ as x approaches 2 is 4. Symbolically, we express this limit as

$$\lim_{x \to 2} f(x) = 4.$$

From this very brief informal look at one limit, let's start to develop an **intuitive definition of the limit**. We can think of the limit of a function at a number a as being the one real number L that the functional values approach as the x-values approach a, provided such a real number L exists. Stated more carefully, we have the following definition:

Definition

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If *all* values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach the number a, then we say that the limit of $f(x)$ as x approaches a is L. (More succinct, as x gets closer to a, $f(x)$ gets closer and stays close to L.) Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L. \tag{2.3}$$

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy.

Problem-Solving Strategy: Evaluating a Limit Using a Table of Functional Values

1. To evaluate $\lim_{x \to a} f(x)$, we begin by completing a table of functional values. We should choose two sets of x-values—one set of values approaching a and less than a, and another set of values approaching a and greater than a. Table 2.1 demonstrates what your tables might look like.

x	$f(x)$		x	$f(x)$
$a - 0.1$	$f(a - 0.1)$		$a + 0.1$	$f(a + 0.1)$
$a - 0.01$	$f(a - 0.01)$		$a + 0.01$	$f(a + 0.01)$
$a - 0.001$	$f(a - 0.001)$		$a + 0.001$	$f(a + 0.001)$
$a - 0.0001$	$f(a - 0.0001)$		$a + 0.0001$	$f(a + 0.0001)$
Use additional values as necessary.			Use additional values as necessary.	

Table 2.1 Table of Functional Values for $\lim_{x \to a} f(x)$

2. Next, let's look at the values in each of the $f(x)$ columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence $f(a - 0.1), f(a - 0.01), f(a - 0.001)., f(a - 0.0001),$ and so on, and $f(a + 0.1), f(a + 0.01), f(a + 0.001), f(a + 0.0001),$ and so on. (*Note*: Although we have chosen the x-values $a \pm 0.1, a \pm 0.01, a \pm 0.001, a \pm 0.0001,$ and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.)
3. If both columns approach a common y-value L, we state $\lim_{x \to a} f(x) = L.$ We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.

4. Using a graphing calculator or computer software that allows us graph functions, we can plot the function $f(x)$, making sure the functional values of $f(x)$ for x-values near a are in our window. We can use the trace feature to move along the graph of the function and watch the y-value readout as the x-values approach a. If the y-values approach L as our x-values approach a from both directions, then $\lim_{x \to a} f(x) = L$. We may need to zoom in on our graph and repeat this process several times.

We apply this Problem-Solving Strategy to compute a limit in **Example 2.4**.

Example 2.4

Evaluating a Limit Using a Table of Functional Values 1

Evaluate $\lim_{x \to 0} \frac{\sin x}{x}$ using a table of functional values.

Solution

We have calculated the values of $f(x) = (\sin x)/x$ for the values of x listed in **Table 2.2**.

x	$\frac{\sin x}{x}$		x	$\frac{\sin x}{x}$
−0.1	0.998334166468		0.1	0.998334166468
−0.01	0.999983333417		0.01	0.999983333417
−0.001	0.999999833333		0.001	0.999999833333
−0.0001	0.999999998333		0.0001	0.999999998333

Table 2.2
Table of Functional Values for $\lim_{x \to 0} \frac{\sin x}{x}$

Note: The values in this table were obtained using a calculator and using all the places given in the calculator output.

As we read down each $\frac{(\sin x)}{x}$ column, we see that the values in each column appear to be approaching one. Thus, it is fairly reasonable to conclude that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. A calculator or computer-generated graph of $f(x) = \frac{(\sin x)}{x}$ would be similar to that shown in **Figure 2.13**, and it confirms our estimate.

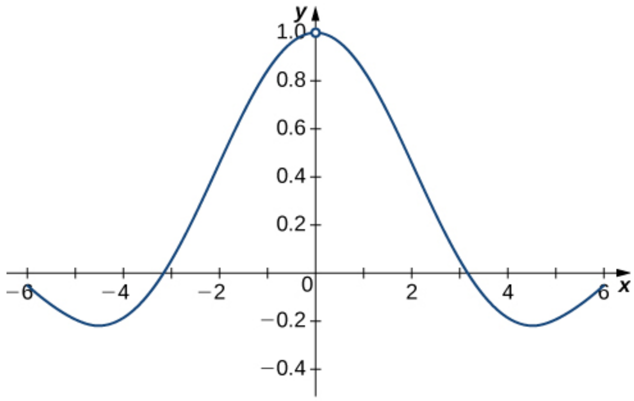


Figure 2.13 The graph of $f(x) = (\sin x)/x$ confirms the estimate from **Table 2.2**.

Example 2.5

Evaluating a Limit Using a Table of Functional Values 2

Evaluate $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$ using a table of functional values.

Solution

As before, we use a table—in this case, **Table 2.3**—to list the values of the function for the given values of x.

x	$\frac{\sqrt{x} - 2}{x - 4}$		x	$\frac{\sqrt{x} - 2}{x - 4}$
3.9	0.251582341869		4.1	0.248456731317
3.99	0.25015644562		4.01	0.24984394501
3.999	0.250015627		4.001	0.249984377
3.9999	0.250001563		4.0001	0.249998438
3.99999	0.25000016		4.00001	0.24999984

Table 2.3
Table of Functional Values for $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

After inspecting this table, we see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25. We conclude that

$\lim\limits_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = 0.25.$ We confirm this estimate using the graph of $f(x) = \frac{\sqrt{x} - 2}{x - 4}$ shown in **Figure 2.14**.

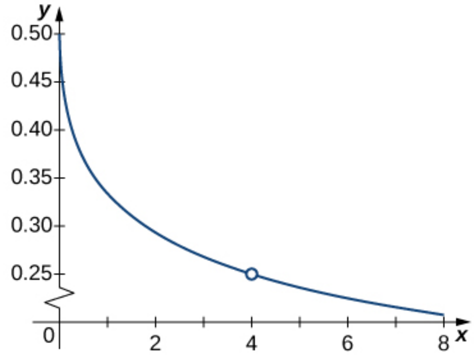


Figure 2.14 The graph of $f(x) = \frac{\sqrt{x} - 2}{x - 4}$ confirms the estimate from **Table 2.3**.

2.4 Estimate $\lim\limits_{x \to 1} \frac{\frac{1}{x} - 1}{x - 1}$ using a table of functional values. Use a graph to confirm your estimate.

At this point, we see from **Example 2.4** and **Example 2.5** that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph as it is to estimate the limit by using a table of functional values. In **Example 2.6**, we evaluate a limit exclusively by looking at a graph rather than by using a table of functional values.

Example 2.6

Evaluating a Limit Using a Graph

For $g(x)$ shown in **Figure 2.15**, evaluate $\lim_{x \to -1} g(x)$.

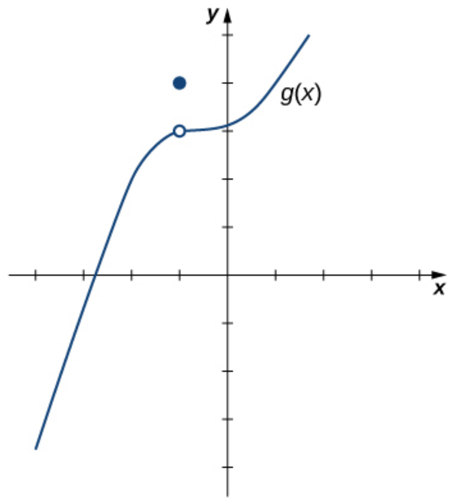


Figure 2.15 The graph of $g(x)$ includes one value not on a smooth curve.

Solution

Despite the fact that $g(-1) = 4$, as the x-values approach -1 from either side, the $g(x)$ values approach 3. Therefore, $\lim_{x \to -1} g(x) = 3$. Note that we can determine this limit without even knowing the algebraic expression of the function.

Based on **Example 2.6**, we make the following observation: It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be different.

2.5 Use the graph of $h(x)$ in **Figure 2.16** to evaluate $\lim_{x \to 2} h(x)$, if possible.

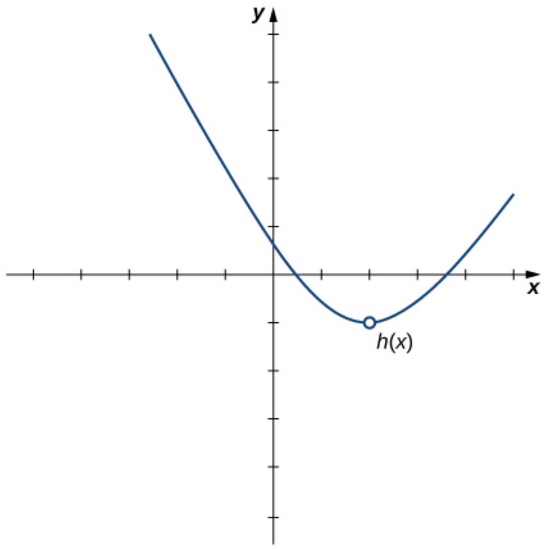


Figure 2.16

Looking at a table of functional values or looking at the graph of a function provides us with useful insight into the value of the limit of a function at a given point. However, these techniques rely too much on guesswork. We eventually need to develop alternative methods of evaluating limits. These new methods are more algebraic in nature and we explore them in the next section; however, at this point we introduce two special limits that are foundational to the techniques to come.

Theorem 2.1: Two Important Limits

Let a be a real number and c be a constant.

i. $$\lim_{x \to a} x = a \quad (2.4)$$

ii. $$\lim_{x \to a} c = c \quad (2.5)$$

We can make the following observations about these two limits.

i. For the first limit, observe that as x approaches a, so does $f(x)$, because $f(x) = x$. Consequently, $\lim_{x \to a} x = a$.

ii. For the second limit, consider **Table 2.4**.

x	$f(x) = c$		x	$f(x) = c$
$a - 0.1$	c		$a + 0.1$	c
$a - 0.01$	c		$a + 0.01$	c
$a - 0.001$	c		$a + 0.001$	c
$a - 0.0001$	c		$a + 0.0001$	c

Table 2.4 Table of Functional Values for $\lim_{x \to a} c = c$

Observe that for all values of x (regardless of whether they are approaching a), the values $f(x)$ remain constant at c. We have no choice but to conclude $\lim_{x \to a} c = c$.

The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

Example 2.7

Evaluating a Limit That Fails to Exist

Evaluate $\lim_{x \to 0} \sin(1/x)$ using a table of values.

Solution

Table 2.5 lists values for the function $\sin(1/x)$ for the given values of x.

x	$\sin\left(\frac{1}{x}\right)$		x	$\sin\left(\frac{1}{x}\right)$
−0.1	0.544021110889		0.1	−0.544021110889
−0.01	0.50636564111		0.01	−0.50636564111
−0.001	−0.8268795405312		0.001	0.826879540532
−0.0001	0.305614388888		0.0001	−0.305614388888
−0.00001	−0.035748797987		0.00001	0.035748797987
−0.000001	0.349993504187		0.000001	−0.349993504187

Table 2.5
Table of Functional Values for $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

After examining the table of functional values, we can see that the y-values do not seem to approach any one single value. It appears the limit does not exist. Before drawing this conclusion, let's take a more systematic approach. Take the following sequence of x-values approaching 0:

$$\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}, \frac{2}{11\pi}, \ldots.$$

The corresponding y-values are

$$1, -1, 1, -1, 1, -1, \ldots.$$

At this point we can indeed conclude that $\lim_{x \to 0} \sin(1/x)$ does not exist. (Mathematicians frequently abbreviate "does not exist" as DNE. Thus, we would write $\lim_{x \to 0} \sin(1/x)$ DNE.) The graph of $f(x) = \sin(1/x)$ is shown in **Figure 2.17** and it gives a clearer picture of the behavior of $\sin(1/x)$ as x approaches 0. You can see that $\sin(1/x)$ oscillates ever more wildly between −1 and 1 as x approaches 0.

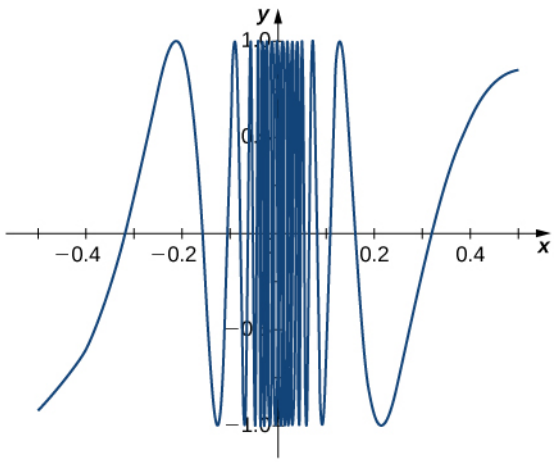


Figure 2.17 The graph of $f(x) = \sin(1/x)$ oscillates rapidly between −1 and 1 as x approaches 0.

2.6 Use a table of functional values to evaluate $\lim_{x \to 2} \frac{|x^2 - 4|}{x - 2}$, if possible.

One-Sided Limits

Sometimes indicating that the limit of a function fails to exist at a point does not provide us with enough information about the behavior of the function at that particular point. To see this, we now revisit the function $g(x) = |x - 2|/(x - 2)$ introduced at the beginning of the section (see **Figure 2.12**(b)). As we pick values of x close to 2, $g(x)$ does not approach a single value, so the limit as x approaches 2 does not exist—that is, $\lim_{x \to 2} g(x)$ DNE. However, this statement alone does not give us a complete picture of the behavior of the function around the x-value 2. To provide a more accurate description, we introduce the idea of a **one-sided limit**. For all values to the left of 2 (or *the negative side of* 2), $g(x) = -1$. Thus, as x approaches 2 from the left, $g(x)$ approaches −1. Mathematically, we say that the limit as x approaches 2 from the left is −1. Symbolically, we express this idea as

$$\lim_{x \to 2^-} g(x) = -1.$$

Similarly, as x approaches 2 from the right (or *from the positive side*), $g(x)$ approaches 1. Symbolically, we express this idea as

$$\lim_{x \to 2^+} g(x) = 1.$$

We can now present an informal definition of one-sided limits.

Definition

We define two types of **one-sided limits**.

Limit from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (c, a), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x < a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the left. Symbolically, we express this idea as

$$\lim_{x \to a^-} f(x) = L. \quad (2.6)$$

Limit from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x) = L. \quad (2.7)$$

Example 2.8

Evaluating One-Sided Limits

For the function $f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ x^2 - 4 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

Solution

We can use tables of functional values again **Table 2.6**. Observe that for values of x less than 2, we use $f(x) = x + 1$ and for values of x greater than 2, we use $f(x) = x^2 - 4$.

x	$f(x) = x + 1$		x	$f(x) = x^2 - 4$
1.9	2.9		2.1	0.41
1.99	2.99		2.01	0.0401
1.999	2.999		2.001	0.004001
1.9999	2.9999		2.0001	0.00040001
1.99999	2.99999		2.00001	0.0000400001

Table 2.6

Table of Functional Values for $f(x) = \begin{cases} x + 1 \text{ if } x < 2 \\ x^2 - 4 \text{ if } x \geq 2 \end{cases}$

Based on this table, we can conclude that a. $\lim_{x \to 2^-} f(x) = 3$ and b. $\lim_{x \to 2^+} f(x) = 0$. Therefore, the (two-sided) limit of $f(x)$ does not exist at $x = 2$. **Figure 2.18** shows a graph of $f(x)$ and reinforces our conclusion about these limits.

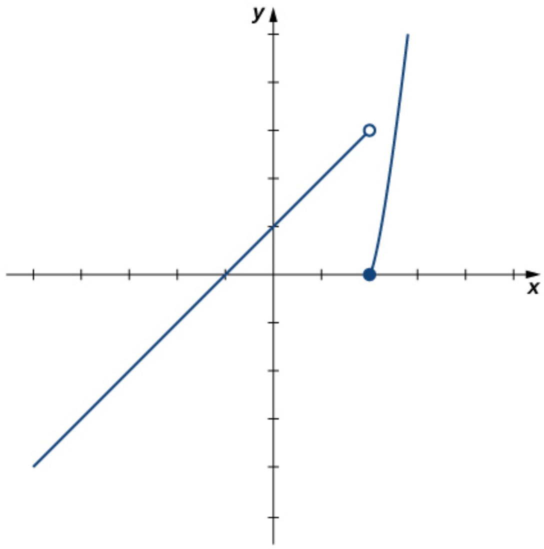


Figure 2.18 The graph of $f(x) = \begin{cases} x + 1 \text{ if } x < 2 \\ x^2 - 4 \text{ if } x \geq 2 \end{cases}$ has a break at $x = 2$.

2.7 Use a table of functional values to estimate the following limits, if possible.

a. $\lim_{x \to 2^-} \frac{|x^2 - 4|}{x - 2}$

b. $\lim_{x \to 2^+} \frac{|x^2 - 4|}{x - 2}$

Let us now consider the relationship between the limit of a function at a point and the limits from the right and left at that point. It seems clear that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function does not exist. These conclusions are summarized in **Relating One-Sided and Two-Sided Limits**.

Theorem 2.2: Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. Then,

$$\lim_{x \to a} f(x) = L. \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Infinite Limits

Evaluating the limit of a function at a point or evaluating the limit of a function from the right and left at a point helps us to characterize the behavior of a function around a given value. As we shall see, we can also describe the behavior of functions that do not have finite limits.

We now turn our attention to $h(x) = 1/(x-2)^2$, the third and final function introduced at the beginning of this section (see **Figure 2.12**(c)). From its graph we see that as the values of x approach 2, the values of $h(x) = 1/(x-2)^2$ become larger and larger and, in fact, become infinite. Mathematically, we say that the limit of $h(x)$ as x approaches 2 is positive infinity. Symbolically, we express this idea as

$$\lim_{x \to 2} h(x) = +\infty.$$

More generally, we define **infinite limits** as follows:

Definition

We define three types of **infinite limits**.

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

i. If the values of $f(x)$ increase without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^-} f(x) = +\infty. \quad \text{(2.8)}$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty. \quad \text{(2.9)}$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

i. If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty. \quad \text{(2.10)}$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty. \quad \text{(2.11)}$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a.

i. If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty. \quad \text{(2.12)}$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty. \quad \text{(2.13)}$$

It is important to understand that when we write statements such as $\lim_{x \to a} f(x) = +\infty$ or $\lim_{x \to a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at a, it must approach a real number L as x approaches a. That said, if, for example, $\lim_{x \to a} f(x) = +\infty$, we always write $\lim_{x \to a} f(x) = +\infty$ rather than $\lim_{x \to a} f(x)$ DNE.

Example 2.9

Recognizing an Infinite Limit

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x$ to confirm your conclusion.

a. $\lim\limits_{x \to 0^-} \frac{1}{x}$

b. $\lim\limits_{x \to 0^+} \frac{1}{x}$

c. $\lim\limits_{x \to 0} \frac{1}{x}$

Solution

Begin by constructing a table of functional values.

x	$\frac{1}{x}$		x	$\frac{1}{x}$
−0.1	−10		0.1	10
−0.01	−100		0.01	100
−0.001	−1000		0.001	1000
−0.0001	−10,000		0.0001	10,000
−0.00001	−100,000		0.00001	100,000
−0.000001	−1,000,000		0.000001	1,000,000

Table 2.7
Table of Functional Values for $f(x) = \frac{1}{x}$

a. The values of $1/x$ decrease without bound as x approaches 0 from the left. We conclude that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

b. The values of $1/x$ increase without bound as x approaches 0 from the right. We conclude that

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

c. Since $\lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim\limits_{x \to 0^+} \frac{1}{x} = +\infty$ have different values, we conclude that

$$\lim_{x \to 0} \frac{1}{x} \text{ DNE.}$$

The graph of $f(x) = 1/x$ in **Figure 2.19** confirms these conclusions.

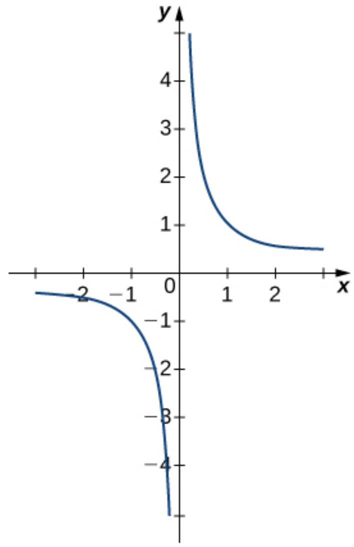


Figure 2.19 The graph of $f(x) = 1/x$ confirms that the limit as x approaches 0 does not exist.

2.8 Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x^2$ to confirm your conclusion.

a. $\lim_{x \to 0^-} \frac{1}{x^2}$

b. $\lim_{x \to 0^+} \frac{1}{x^2}$

c. $\lim_{x \to 0} \frac{1}{x^2}$

It is useful to point out that functions of the form $f(x) = 1/(x - a)^n$, where n is a positive integer, have infinite limits as x approaches a from either the left or right (**Figure 2.20**). These limits are summarized in **Infinite Limits from Positive Integers**.

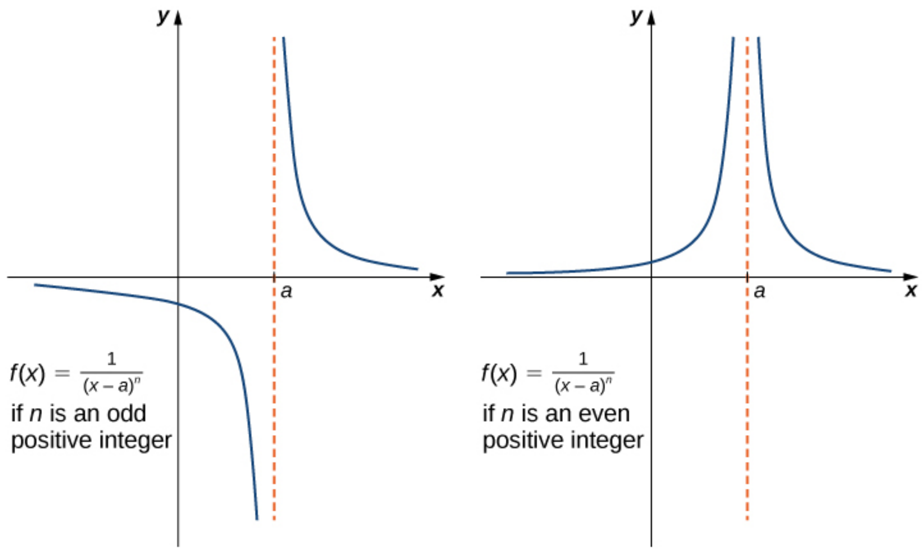


Figure 2.20 The function $f(x) = 1/(x - a)^n$ has infinite limits at a.

Theorem 2.3: Infinite Limits from Positive Integers

If n is a positive even integer, then

$$\lim_{x \to a} \frac{1}{(x-a)^n} = +\infty.$$

If n is a positive odd integer, then

$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = +\infty$$

and

$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty.$$

We should also point out that in the graphs of $f(x) = 1/(x-a)^n$, points on the graph having x-coordinates very near to a are very close to the vertical line $x = a$. That is, as x approaches a, the points on the graph of $f(x)$ are closer to the line $x = a$. The line $x = a$ is called a **vertical asymptote** of the graph. We formally define a vertical asymptote as follows:

Definition

Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a **vertical asymptote** of $f(x)$.

$$\begin{aligned} \lim_{x \to a^-} f(x) &= +\infty \text{ or } -\infty \\ \lim_{x \to a^+} f(x) &= +\infty \text{ or } -\infty \\ &\text{or} \\ \lim_{x \to a} f(x) &= +\infty \text{ or } -\infty \end{aligned}$$

Example 2.10

Finding a Vertical Asymptote

Evaluate each of the following limits using **Infinite Limits from Positive Integers**. Identify any vertical asymptotes of the function $f(x) = 1/(x+3)^4$.

a. $\lim_{x \to -3^-} \frac{1}{(x+3)^4}$

b. $\lim_{x \to -3^+} \frac{1}{(x+3)^4}$

c. $\lim_{x \to -3} \frac{1}{(x+3)^4}$

Solution

We can use **Infinite Limits from Positive Integers** directly.

a. $\lim_{x \to -3^-} \frac{1}{(x+3)^4} = +\infty$

b. $\lim_{x \to -3^+} \frac{1}{(x+3)^4} = +\infty$

c. $\lim_{x \to -3} \frac{1}{(x+3)^4} = +\infty$

The function $f(x) = 1/(x+3)^4$ has a vertical asymptote of $x = -3$.

2.9 Evaluate each of the following limits. Identify any vertical asymptotes of the function $f(x) = \frac{1}{(x-2)^3}$.

a. $\lim_{x \to 2^-} \frac{1}{(x-2)^3}$

b. $\lim_{x \to 2^+} \frac{1}{(x-2)^3}$

c. $\lim_{x \to 2} \frac{1}{(x-2)^3}$

In the next example we put our knowledge of various types of limits to use to analyze the behavior of a function at several different points.

Example 2.11

Behavior of a Function at Different Points

Use the graph of $f(x)$ in **Figure 2.21** to determine each of the following values:

a. $\lim_{x \to -4^-} f(x)$; $\lim_{x \to -4^+} f(x)$; $\lim_{x \to -4} f(x)$; $f(-4)$

b. $\lim_{x \to -2^-} f(x)$; $\lim_{x \to -2^+} f(x)$; $\lim_{x \to -2} f(x)$; $f(-2)$

c. $\lim_{x \to 1^-} f(x)$; $\lim_{x \to 1^+} f(x)$; $\lim_{x \to 1} f(x)$; $f(1)$

d. $\lim_{x \to 3^-} f(x)$; $\lim_{x \to 3^+} f(x)$; $\lim_{x \to 3} f(x)$; $f(3)$

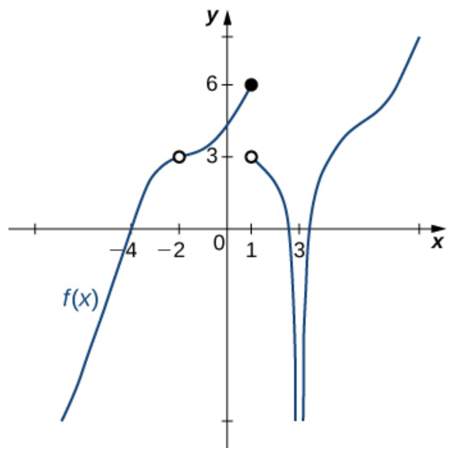


Figure 2.21 The graph shows $f(x)$.

Solution

Using **Infinite Limits from Positive Integers** and the graph for reference, we arrive at the following values:

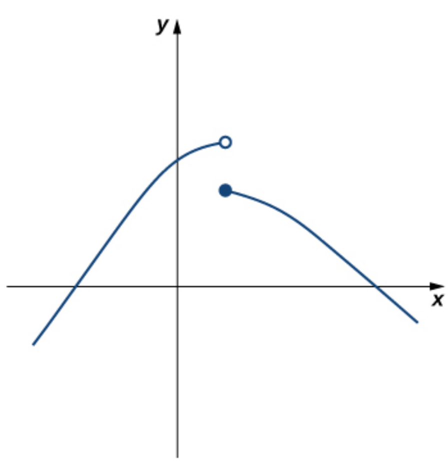
a. $\lim_{x \to -4^-} f(x) = 0$; $\lim_{x \to -4^+} f(x) = 0$; $\lim_{x \to -4} f(x) = 0$; $f(-4) = 0$

b. $\lim_{x \to -2^-} f(x) = 3.$; $\lim_{x \to -2^+} f(x) = 3$; $\lim_{x \to -2} f(x) = 3$; $f(-2)$ is undefined

c. $\lim_{x \to 1^-} f(x) = 6$; $\lim_{x \to 1^+} f(x) = 3$; $\lim_{x \to 1} f(x)$ DNE; $f(1) = 6$

d. $\lim_{x \to 3^-} f(x) = -\infty$; $\lim_{x \to 3^+} f(x) = -\infty$; $\lim_{x \to 3} f(x) = -\infty$; $f(3)$ is undefined

2.10 Evaluate $\lim_{x \to 1} f(x)$ for $f(x)$ shown here:



Example 2.12

Chapter Opener: Einstein's Equation

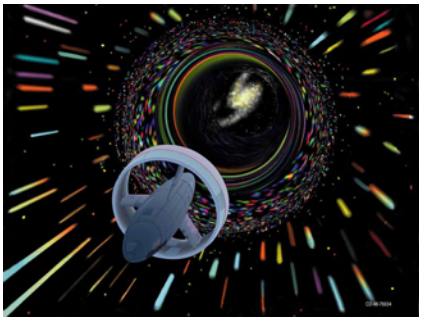


Figure 2.22 (credit: NASA)

In the chapter opener we mentioned briefly how Albert Einstein showed that a limit exists to how fast any object can travel. Given Einstein's equation for the mass of a moving object, what is the value of this bound?

Solution

Our starting point is Einstein's equation for the mass of a moving object,

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. To see how the mass changes at high speeds, we can graph the ratio of masses m/m_0 as a function of the ratio of speeds, v/c (Figure 2.23).

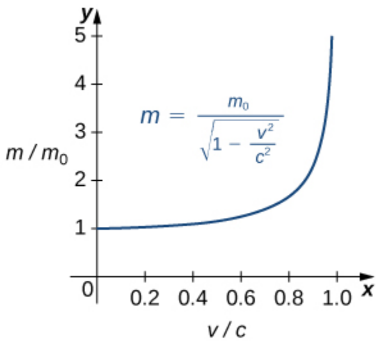


Figure 2.23 This graph shows the ratio of masses as a function of the ratio of speeds in Einstein's equation for the mass of a moving object.

We can see that as the ratio of speeds approaches 1—that is, as the speed of the object approaches the speed of light—the ratio of masses increases without bound. In other words, the function has a vertical asymptote at $v/c = 1$. We can try a few values of this ratio to test this idea.

$\frac{v}{c}$	$\sqrt{1-\frac{v^2}{c^2}}$	$\frac{m}{m_0}$
0.99	0.1411	7.089
0.999	0.0447	22.37
0.9999	0.0141	70.71

Table 2.8
Ratio of Masses and Speeds for a Moving Object

Thus, according to **Table 2.8**, if an object with mass 100 kg is traveling at 0.9999c, its mass becomes 7071 kg. Since no object can have an infinite mass, we conclude that no object can travel at or more than the speed of light.